

Chapter 4

Boolean Algebra

SKEE1223 Digital Electronics

Mun'im/Arif

FKE, Universiti Teknologi Malaysia

October 30, 2015

Overview

- 1 Boolean Theorems
- 2 Expressing Logic Circuits
- 3 Analysis of Circuits
- 4 Canonical Forms
- 5 Boolean Simplification

Boolean Axioms

- Axiom (or postulate): mathematical statement without proof

- Axiom 1:

$$A = 0 \text{ if } A \neq 1$$

$$A = 1 \text{ if } A \neq 0$$

- Axiom 2:

$$A = 0 \text{ if } \bar{A} \neq 1$$

$$A = 1 \text{ if } \bar{A} \neq 0$$

- Axiom 3:

$$0 \cdot 0 = 0$$

$$1 + 1 = 1$$

$$1 \cdot 1 = 1$$

$$0 + 0 = 0$$

$$0 \cdot 1 = 1 \cdot 0 = 0$$

$$0 + 1 = 1 + 0 = 1$$

Single Variable Theorems

Theorem	AND Form	OR Form
Annulment	$A \cdot 0 = 0$	$A + 1 = 1$
Identity	$A \cdot 1 = A$	$A + 0 = A$
Idempotency	$A \cdot A = A$	$A + A = A$
Complementation	$A \cdot \bar{A} = 0$	$A + \bar{A} = 1$
Involution	$\overline{\bar{A}} = A$	

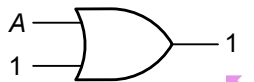
Annulment Theorem



$$A \cdot 0 = 0$$

A	B	F
0	0	0
0	1	0
1	0	0
1	1	1

The first three rows of the truth table are highlighted with a light blue background and a dotted border, indicating that the output is 0 for all combinations of A and B.



$$A + 1 = 1$$

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

The last three rows of the truth table are highlighted with a light blue background and a dotted border, indicating that the output is 1 for all combinations of A and B.

Identity Theorem



$$A \cdot 1 = A$$

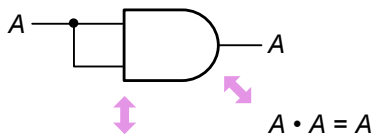
A	B	F
0	0	0
0	1	0
1	0	0
1	1	1



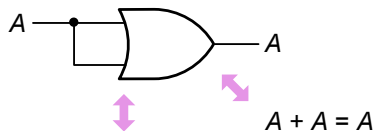
$$A + 0 = A$$

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

Idempotency Theorem

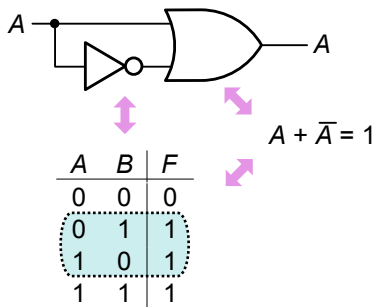
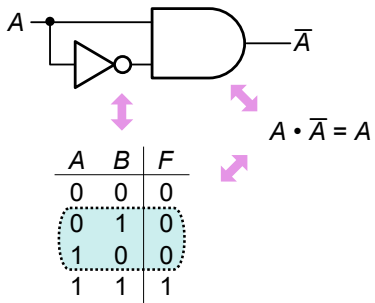


A	B	F
0	0	0
0	1	0
1	0	0
1	1	1



A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

Complements Theorem



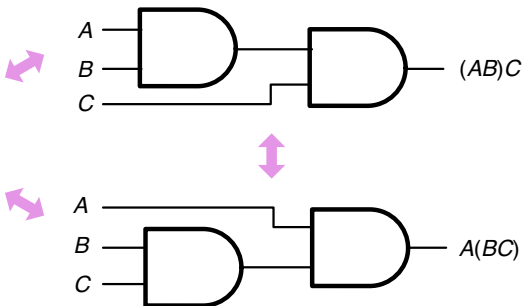
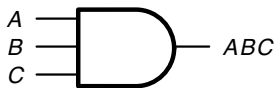
Multivariable Theorems

Theorem	Function	Dual
Commutation	$AB = BA$	$A + B = B + A$
Association	$(AB)C = A(BC)$ $= ABC$	$(A + B) + C = A + (B + C)$ $= A + B + C$
Distribution	$A(B + C) = AB + AC$	$A + BC = (A + B)(A + C)$
De Morgan's	$\overline{A + B} = \overline{A} \cdot \overline{B}$	$\overline{A \cdot B} = \overline{A} + \overline{B}$

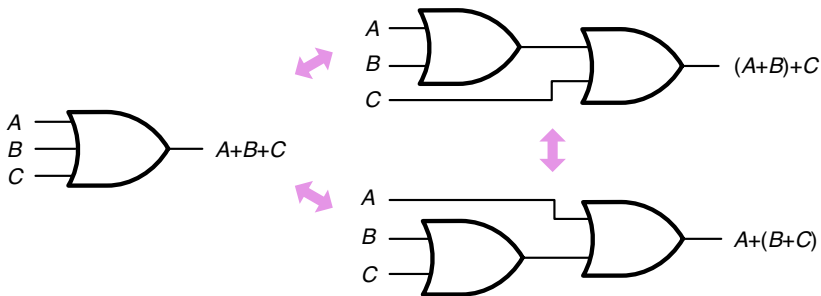
Multivariable Theorems: Commutation



Multivariable Theorems: Association



Multivariable Theorems: Association



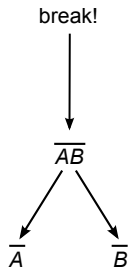
Simplification Theorems

Theorem	Function	Dual
Uniting	$AB + A\bar{B} = A$	$(A + B)(A + \bar{B}) = A$
Absorption	$A + AB = A$	$A(A + B) = A$
Adsorption	$A + \bar{A}B = A + B$	$(A + \bar{B})B = AB$
Consensus	$AB + \bar{A}C$ $= AB + BC + \bar{A}C$	$(A + B)(\bar{A} + C)$ $= (A + B)(B + C)(\bar{A} + C)$
Multiplying	$(A + B)(\bar{A} + C)$ $= AC + \bar{A}B$	$AB + \bar{A}C$ $= (A + C)(\bar{A} + B)$
De Morgan's	$\overline{A + B} = \bar{A} \cdot \bar{B}$	$\overline{A \cdot B} = \bar{A} + \bar{B}$

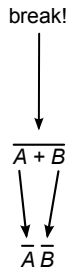
De Morgan's Theorem

- Useful for simplifying inverted variables:

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$



$$\overline{A + B} = \overline{A} \cdot \overline{B}$$



Proofs

- Prove that $A + AB = A$.
- Prove that $A \cdot (A + B) = A$.
- Prove that $A + \overline{A}B = A + B$.
- Prove that $(A + B)(A + C) = A + BC$.

Forms of Expression

Basic Rules:

- Connected gates form a **logic network**
- An output can be connected to 1 or more inputs
- An input can only be connected to 1 output.
- Most common is AND-OR network

Two forms of expression

- Sum-of-Products (SOP)
- Product-of-Sums (POS).

POS to SOP

Easy

Just multiply through and simplify.

Example:

Convert $(A + C)(B + \bar{C})$ to its SOP equivalent.

$$\begin{aligned}F &= (A + C)(B + \bar{C}) \\&= AB + A\bar{C} + BC + C\bar{C} \\&= AB + A\bar{C} + BC + 0 \\&= AB + A\bar{C} + BC\end{aligned}$$

SOP to POS

Apply DeMorgan's twice.

Example:

Convert $F = A\bar{B} + B\bar{C}$ to its POS equivalent.

- 1 Apply DeMorgan's law once to get the **function's inverse**:

$$\begin{aligned}\bar{F} &= \overline{A\bar{B} + B\bar{C}} \\ &= \overline{A\bar{B}} \cdot \overline{B\bar{C}} \\ &= (\bar{A} + B)(\bar{B} + C) \\ &= \bar{A}\bar{B} + \bar{A}C + B\bar{B} + BC \\ &= \bar{A}\bar{B} + \bar{A}C + BC\end{aligned}$$

SOP to POS

2 Apply DeMorgan's law again:

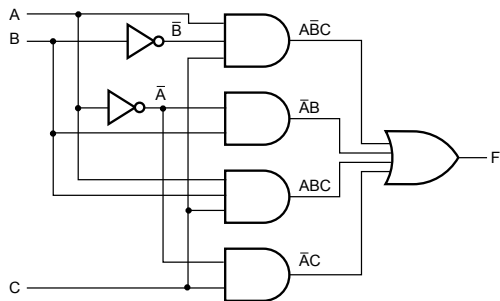
$$\begin{aligned}F &= \overline{\overline{F}} \\ &= \overline{\overline{A\overline{B}} + \overline{A\overline{C}} + \overline{BC}} \\ &= \overline{\overline{A\overline{B}}} \cdot \overline{\overline{A\overline{C}}} \cdot \overline{\overline{BC}} \\ &= (A + B)(A + \overline{C})(\overline{B} + \overline{C})\end{aligned}$$

Describing Logic Circuits Algebraically

Rules to build an AND-OR network:

- 1 Each AND operation requires one AND gate
- 2 Each OR operation requires one OR gate
- 3 Each complemented variable requires a NOT gate.

Describing Logic Circuits Algebraically



$$F = \bar{A}\bar{B}C + \bar{A}B + ABC + \bar{A}C$$

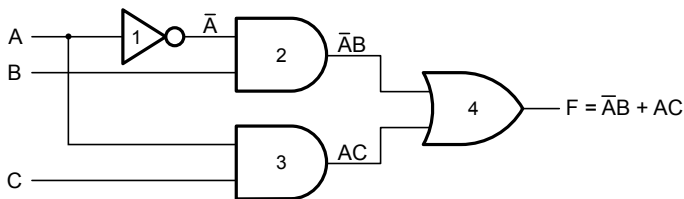
Analysis:

- 1 3 variables
- 2 4 product terms
- 3 10 literals

Analysis of Circuits

Steps for analyzing a circuit:

- 1** The inputs to all gates must have a variable name.
- 2** Write the expression performed by each gate at its output. Do it from left to right.
- 3** The output of the last gate is the network equation



Logic circuit the the expression $F = \bar{A}B + AC$.

Analysis of Circuits

Truth table for $F = \overline{A}B + BC$

A	B	C	\overline{A}	$\overline{A}B$	$\overline{A}C$	\overline{F}
0	0	0	1	0	0	0
0	0	1	1	0	0	0
0	1	0	1	1	0	1
0	1	1	1	1	0	1
1	0	0	0	0	0	0
1	0	1	0	0	1	1
1	1	0	0	0	0	0
1	1	1	0	0	1	1

What & Why Canonical Forms

- Canonical form = standard form
- Useful for comparing seemingly different functions
- Unique for each function
- Two versions:
 - SOP
 - POS

Minterms & Maxterms

Must know Minterms & Maxterms to get canonical form.

- Minterm:

- AND term with every variable present in either true or complemented form.

- Maxterm:

- OR term with every variable present in either true or complemented form.

Minterms for 2-variable Function

A	B	Minterm	Maxterm
0	0	$m_0 = \bar{A}\bar{B}$	$M_0 = A + B$
0	1	$m_1 = \bar{A}B$	$M_1 = A + \bar{B}$
1	0	$m_2 = A\bar{B}$	$M_2 = \bar{A} + B$
1	1	$m_3 = AB$	$M_3 = \bar{A} + \bar{B}$

Important:

- In a minterm or maxterm, each variable appears once, either as the variable itself or as the inverse
- For minterms:
 - 1 means the variable is not complemented and
 - 0 means the variable is complemented.
- For maxterms:
 - 0 means the variable is not complemented and
 - 1 means the variable is complemented.

Minterms for 3-variable Function

A	B	C	Minterm	Maxterm
0	0	0	$m_0 = \bar{A}\bar{B}\bar{C}$	$M_0 = A + B + C$
0	0	1	$m_1 = \bar{A}\bar{B}C$	$M_1 = A + B + \bar{C}$
0	1	0	$m_2 = \bar{A}B\bar{C}$	$M_2 = A + \bar{B} + C$
0	1	1	$m_3 = \bar{A}BC$	$M_3 = A + \bar{B} + \bar{C}$
1	0	0	$m_4 = A\bar{B}\bar{C}$	$M_4 = \bar{A} + B + C$
1	0	1	$m_5 = A\bar{B}C$	$M_5 = \bar{A} + B + \bar{C}$
1	1	0	$m_6 = AB\bar{C}$	$M_6 = \bar{A} + \bar{B} + C$
1	1	1	$m_7 = ABC$	$M_7 = \bar{A} + \bar{B} + \bar{C}$

Canonical SOP

A	B	C	F	
0	0	0	1	$\rightarrow \bar{A}\bar{B}\bar{C} \rightarrow m_0$
0	0	1	0	
0	1	0	1	$\rightarrow \bar{A}B\bar{C} \rightarrow m_2$
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	1	$\rightarrow AB\bar{C} \rightarrow m_6$
1	1	1	1	$\rightarrow ABC \rightarrow m_7$

$$\begin{aligned}
 \therefore F(A, B, C) &= \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + AB\bar{C} + ABC \\
 &= m_0 + m_2 + m_6 + m_7 \\
 &= \sum m(0, 2, 6, 7)
 \end{aligned}$$

Canonical From Non-Canonical SOP

Expand terms using $A + \bar{A} = 1$.

$$\begin{aligned} F &= AB(C + \bar{C}) + A\bar{C}(B + \bar{B}) + BC(A + \bar{A}) \\ &= ABC + AB\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + ABC + \bar{A}BC \\ &= \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + ABC \\ &= m_3 + m_4 + m_6 + m_7 \\ &= \sum m(3, 4, 6, 7) \end{aligned}$$

Canonical POS

A	B	C	F	
0	0	0	1	
0	0	1	0	$\rightarrow A + B + \bar{C} \rightarrow M_1$
0	1	0	1	
0	1	1	0	$\rightarrow A + \bar{B} + \bar{C} \rightarrow M_3$
1	0	0	0	$\rightarrow \bar{A} + B + C \rightarrow M_4$
1	0	1	0	$\rightarrow \bar{A} + B + \bar{C} \rightarrow M_5$
1	1	0	1	
1	1	1	1	

$$\begin{aligned}
 \therefore F(A, B, C) &= (A + B + \bar{C})(A + \bar{B} + \bar{C})(\bar{A} + B + C)(\bar{A} + B + \bar{C}) \\
 &= M_1 M_3 M_4 M_5 \\
 &= \prod M(1, 3, 4, 5)
 \end{aligned}$$

Simplification Procedure

- 1 Convert to SOP (sum of products) form.

$$(A + B)(C + D) \rightarrow AC + AD + BC + BD$$

- 2 Check for exact duplicates and drop.

$$A\bar{B} + BC + A\bar{B} \rightarrow A\bar{B} + BC$$

- 3 Check for null values and drop.

$$\begin{aligned} A(\bar{A} + B) &\rightarrow A\bar{A} + AB \\ &\rightarrow AB \end{aligned}$$

Simplification Procedure

- 4 See if one of the items is entirely contained in another. If it is, factor it out.

$$\begin{aligned}AB + ABC &\rightarrow AB(1 + C) \\ &\rightarrow AB\end{aligned}$$

- 5 See if two of the terms are different by only a NOT function. If it is, factor out all the similar terms.

$$\begin{aligned}ABC + ABC\bar{C} &\rightarrow AB(C + \bar{C}) \\ &\rightarrow AB\end{aligned}$$

Worked Example:

- Simplify $f = \overline{A}B + A\overline{B}C + BC$.

Worked Example:

- Simplify $f = (X + Y)(X + \bar{Y})(+Z)$.

Worked Example:

- Simplify $f = \overline{\overline{A + BC} + \overline{AB}}$.

Worked Example:

- Simplify $F = AB + \bar{A}C + BC$.